Quantization in large scale for the description of planetary orbits in a mass independent approach of celestial bodies.

Purposes:

Mathematical formulations which are applied to the quantum theory for the atomic problem can be relevant to treat some aspects that involve the issue of planetary orbits. In the case of the solar system it will be shown that the mean planetary distances and revolution periods can be determined starting from a single parameter given by the mean distance between Mercury and the sun, in an approach independent of the masses of the bodies in this system. In the case of binary star systems the formulation of the quantum theory for the molecular problem can provide significant results as well.

Begining: the melody of the Cosmos

The immense advancement of knowledge in the study of atomic systems have provided for the mankind access to numerous relevant findings, especially those related to the development of the revolutionary technologies of informatics and nanotechnology. However, the principles that govern the universe of the atom, in other words, the quantum nature of "the infinitely small", had their origins in the beginnings of the thought about the regency of mathematics in the workings of Nature. This thought appeared with Pythagoras and his School in old Greece, from a poetic inference trying to find mathematical relations which could describe natural phenomena, the first of this attempts consisting in establishing a relationship between simple numeric proportions and musical intervals. In this way, ratios between small integers were applied in an effective manner to build the musical scale produced by a string instrument. The pythagoreans later have adopted this notion to extend these ideas about musical harmony and mathematics to the movement of the planets: in a similar way ratios between small integers that rule the musical notes could rule the configuration of celestial bodies in the solar system. So, in the Ancient Greece Pythagoric cosmology, in a geocentric system, the sun and the planets, each in its own orbit, could be described as a musical instrument generating a melodic set of divine sounds expressed by the harmony of the movements of celestial bodies. In 1618, more than 2000 years later, the german astronomer Johannes Kepler resurrected this pythagorean idea about the harmony of the celestial bodies in his book "Harmonice Mundi" (World's Harmony). Combining the orbital velocities of the different planets in a heliocentric system, the astronomer inferred that the ratios between these velocities could be compared to the ratio between the integers obtained in the musical scales.

Planetary distances and Integers

The use of numerology has always fascinated and arouses curiosity among astronomers who aimed to estimate the distances between the sun and the planets. In reality the integers played an important role in the discovery of the planets Uranus (in 1781), Neptune (1846), the Asteroid Belt (1801) between the planets Mars and Jupiter and the asteroid Quiron (1978) between the planets Saturn and Uranus.

If we consider initially the Titus-Bode empiric law, established in 1778 to estimate the planetary distances:

$D = 0.4 + (0.3) \times 2^n$ AU,

where D designates the distance to the n-th planet in Astronomical Unit (AU), being equal to the distance between Earth and the sun. For Mercury we must take D=0.4 therefore resulting this way for Venus (n=0), D=0.7; for Earth (n=1), D= 1.0; for Mars (n=2), D=1.6 and so forth as shown in Table 1.

Table I – Titus-Bode Law Illustration.

Planets	*Calculated Distance (AU)*	*Observed Distance (AU)*
Mercury	*0.4*	*0.3*
Venus	*0.4 + 0.3×1=0.7*	*0.7*
Earth	*0.4 + 0.3×2=1.0*	*1.0*
Mars	*0.4 + 0.3×4=1.6*	*1.5*
–	*0.4 + 0.3×8=2.8*	–
Jupiter	*0.4 + 0.3×16=5.2*	*5.2*
Saturn	*0.4 + 0.3×32=10.0*	*9.6*
Uranus	*0.4 + 0.3×64=19.6*	*19.2*
Neptune	*0.4 + 0.3×128=38.8*	*30.1*

Pluto	*0.4 +0.3x 256=77.2*	*39.5*

By Table I we can observe that the empirical relationship given by the Titus-Bode Law provides precise distances to the interior planets, but starts to show itself imprecise for Saturn, Uranus and Neptune, being unable to predict Pluto's orbit.

It is worth mentioning that when the Law was stated in 1778, no planet other than Saturn was known. The discovery of Uranus in 1781, whose distance in relation to the Sun predicted empirically in a good agreement in respect to its real distance, raised question amongst astronomers in respect to the gap correspondent to the algorism 2.8 as presented in Table I. It could then be found a new planet of a small size located in this position in relation to the sun that was still not discovered at the time. And in fact in the period comprised between 1801 and 1807 the existence of these "small planets" was demonstrated, and where called afterwards "asteroids", with mean distances of 2.8 AU, as indicated by the Titus-Bode Law.

In its turn, the asteroid Quiron was predicted by the Italian astronomer Giuseppe Armellini who by formulating a new empirical law in 1922 indicated the existence of a planet precisely between Saturn and Uranus. In 1978 the North American astronomer Charles T Kowal discovered the asteroid Quiron at a distance from the Sun such as the one predicted based on Armellini's empirical law expressed by the following mathematical relation:

$$D = 1.53^n,$$

where D (in AU), represents the mean distance between the planets and the sun and is given in function of the values of n ranging from $n = -2$ for Mercury; $n = -1$ for Venus; $n = 0$ for the Earth; $n = 1$ for Mars and so forth as shown in Table II:

Table II – Armellini's Law Illustration.

Planets	$D=1.53^n$	*Calculated Distances (AU)*	*Observed Distances (AU)*
Mercury	1.53^{-2}	*0.43*	*0.39*
Venus	1.53^{-1}	*0.65*	*0.72*
Earth	1.53^{0}	*1.00*	*1.00*

Mars	*1.53^1*	*1.53*	*1.52*
Vesta	*1.53^2*	*2.34*	*2.36*
Camilla	*1.53^3*	*3.58*	*3.48*
Jupiter	*1.53^4*	*5.48*	*5.20*
Saturn	*1.53^5*	*8.38*	*9.55*
Quiron	*1.53^6*	*12.83*	*13.87*
Uranus	*1.53^7*	*19.63*	*19.20*
Neptune	*1.53^8*	*30.03*	*30.11*
Pluto	*1.53^9*	*45.94*	*39.52*

The asteroids Vesta and Camilla are represented in Table II, respectively, the interior and exterior limits of the main asteroid belt with around the two thousand asteroids, which existence between Mars and Jupiter was predicted in 1766 by the Titius-Bode Law. Similarly, the Armellini Law, besides contemplating the distances of the known planets and asteroids, predicted the existence of another asteroid situated between Saturn and Uranus. However, this law shown itself unsatisfactory to determine the observed distance for the planet Pluto.

Goals of the present investigation: Approaches based on the quantum theory for the description of planetary orbits.

1 – The atomic model of Bohr and the Solar System [1]

It will be shown, in a similar procedure as adopted by Niels Bohr to describe the possible orbits of the single electron around the nucleus of the hydrogen atom, that it is possible to combine the mathematical equation that expresses Newton's Second Law for the orbital movement of the planets with the mathematical equation that expresses Bohr quantization rule to obtain an expression for the mean planetary distances to the sun's center. The resulting expressions allows to state that in a general way, the ratio between the mean planetary distances must obey the ratio between the square of the integers associated to those orbits, and to the revolution periods the ratio between the cubes of these same integers.

2 – A similar equation to the one of Schrödinger for the description of Solar System's orbits [2-8]

By bringing back the ideas of another physicist who was fundamental to the creation of the field of Quantum Mechanics - Erwin Schrödinger - by using an equation similar to the one he postulated for studying the universe of the atomic and subatomic worlds, was applied in this approach to describe the solar system in a plane involving an attractive field placed in the center of the system associated to a body with mass *M*, allowing these studies to obtain the following group of results:

- Mean planetary radius to all Terrestrial planets (Mercury, Venus, Earth and Mars).
- Mean planetary radius to all the Jovian planets (Jupiter, Saturn, Uranus and Neptune).
- Mean planetary radius to the dwarf planets - Pluto - and those recently discovered (Makemake, Haumea and Eris).
- Mean radius to the main probability regions to find bodies in the Hungarian Asteroid Belt and in the Main Asteroid Belt.
- Mean radius to all the Centaur asteroids and most of the trans-neptunian.

3 – Molecular orbital method applied to the treatment of an extrasolar binary star system [7-8]

A similar approach to the one used for the calculation of equilibrium distances in a diatomic molecular systems was applied to estimate the mean distances of separation between binary stars in the extrasolar system HD 188753 Cygni. Based on the functions of tridimensional waves that describe the atomic orbit 1s for the hydrogen atom, and taking a_0 = 0.387 AU, i.e, the mean planetary radius of Mercury, the variational method foresees a "equilibrium distance" for the two stars of approximately 1.0 AU. This result is similar to the separation observed of 0.66 AU.

Recent observations have been done to look into the formation geometry of protoplanetary disks around binary stars also allowing to describe the formation of circumbinary and circumstellar debris around these systems. By applying a molecular quantum mechanics model for the description of eletronic cloud distributions for the molecular system H_2^+ the results show that the description of the formation geometries of protoplanetary disks in function of the separation distance between the binary stars is similar to the distribution of the eletronic cloud around the two protons of the H_2^+ molecular ion in function of their separation distances.

1. Application of the Bohr model to the Solar System

This model that was applied successfully to the hydrogen atom consists of a nucleus with a positive charge and an electron with negative charge moving around this nucleus in a circular orbit. In a similar fashion to a miniature of the Solar System only a few specific orbits would be allowed. The intrinsically discontinuous nature of the atomic world was imposed by the condition that a physical quantity associated to the electron movement in an orbit, designated of angular moment (*L*) of the electron (the product of its mass *m* by its orbital velocity *v* and its distance *r* to the atomic nucleus), was imposed be written by multiples of integers *n* of the Planck's constant h or "quantum of action" divided by *2π.*

$$L = n\frac{h}{2\pi}, \quad onde \;\; n = 1,2,3...;$$

designated quantization rule of Bohr.

Each electron orbit around the hydrogen atom nucleus, in the context of the Bohr model, is therefore labeled by the integer *n*, starting with *n=1* which characterizes the fundamental state of this atom, therefore, the unit is associated to the first allowed orbit, that one closest to the nuclear region.

In an analogous formulation, for the Solar System, in which the intensity of the gravitational and centripetal force are equal:

$$m_p v^2/r = GM_S m_p/r^2$$

According to the De Broglie's wave-particle duality principle:

$$\lambda = g^*/p = g^*/m_p v$$

where λ corresponds to the wave lenght associated to the linear moment *p* of each planet and *g** is a constant equivalent to Planck's constant.

Assuming circular orbits for the planets the stationary wave condition is given by:

$$2\pi r = n\lambda \qquad (n = 1,2,3,...)$$

These two last equations show that for each orbit the intensity of the angular moment can be described as:

$$L = m_p v r = n g^* / 2\pi$$

which is simillar to the semi-classic condition imposed by Bohr for the quantization of the angular moment ($L = nh/2\pi$).

The set of equations above shows that:

$$r = (n^2 g^{*2}) / (4\pi^2 G M_s m_p^2)$$

and the orbital velocity given by:

$$v = 2\pi G M_s m_p / n g^*$$

The revolution period T for each planet can then be determined:

$$T = n^3 [(g^{*3}) / (4\pi G M_s^2 m_p^2)$$

The second and third Kepler laws for the planetary orbits can be easily verified based on the equations for *r* and *T*.

If n_i and n_j are numbers corresponding to any two orbits *i* and *j*, respectively, then:

$$r_i = (n_i^2 g^{*2})/(4\pi^2 GM_s m_p^2)$$

and

$$r_j = (n_j^2 g^{*2})/(4\pi^2 GM_s m_p^2)$$

so,

$$r_i/r_j = n_i^2/n_j^2$$

For the revolution periods *T*

$$T_i = n_i^3 [(g^{*3})/(4\pi GM_s^2 m_p^2)$$

and

$$T_j = n_j^3 [(g^{*3})/(4\pi GM_s^2 m_p^2)$$

so,

$$T_i/T_j = n_i^3/n_j^3$$

The ratio between the mean distances and periods like the ones given above suggest the following principle:

The ratio between the mean planetary distances obeys the ratio between the square of the integers associated to their respective orbits, and for the periods, the ratio between the cubes of these same integers.

We have then the following expressions of recurrence for the planetary orbits:

$$r_{i+1} = r_i [(n_{i+1})/(n_i)]^2$$

and

$$T_{i+1} = T_i [(n_{i+1})/(n_i)]^3$$

Table III. Mean planetary radius and periods calculated directly from the recurrence expressions.

n	$r_{i+1} = r_i[(n_i+1)/n_i)]^2$	$T_{i=1}$= Ti $[(n_i+1) /(n_i)^3$
1	r_1=0.387	T_1= 0.241
2	$r_2=r_1(2/1)^2$ = 1.548	T_2= $T_1(2/1)^3$=1.928
3	$r_3=r_2(3/2)^2$ = 3.484	$T_3=T_2(3/2)^3$=6.507
4	$r_4=r_3(4/3)^2$ = 6.194	$T_4=T_3(4/3)^3$=15.424

5	$r_5=r_4(5/4)^2 = 9.678$	$T_5=T_4(5/4)^3=30.125$
6	$r_6=r_5(6/5)^2 = 13.936$	$T_6=T_5(6/5)^3=52.056$
7	$r_7=r_6(7/6)^2 = 18.968$	$T_7=T_6(7/6)^3=82.663$
8	$r_8=r_7(8/7)^2 = 24.774$	$T_8=T_7(8/7)^3=123.392$
9	$r_9=r_8(9/8)^2 = 31.355$	$T_9=T_8(9/8)^3=175.689$
10	$r_{10}=r_9(10/9)^2 = 38.710$	$T_{10}=T_9(10/9)^3=241.0$
11	$r_{11}=r_{10}(11/10)^2 = 46.839$	$T_{11}=T_{10}(11/10)^3=320.771$
12	$r_{12}=r_{11}(12/11)^2 = 55.742$	$T_{12}=T_{11}(12/11)^3=416.448$
13	$r_{13}=r_{12}(13/12)^2 = 65.419$	$T_{13}=T_{12}(13/12)^3=529.447$

In Table III it can be observed that the mean radius for the planets Venus and Earth as well as the mean radius associated to the asteroid that describes the interior orbit of the Asteroid Belt and those of the Hungarian Belt could not be verified.

A procedure for the calculation of the mean distances of the planets Venus and Earth and of the asteroid associated to the interior orbit of the Main Asteroid Belt and those of the Hungarian Belt

These orbits are empirically calculated establishing a class of numbers *m*, associated to the value of the integer *n* corresponding to a given orbit, ranging from zero to *n*.

For example: consider $n = 2$ with m = *0*, *1, 2* and through the pair ($n = 2; m = 2$) the orbits associated to the pairs ($n = 2; m = 1$) and ($n = 2; m = 0$) can be determined from the following calculations:

$$r_{2,1} = r_{2,2}[(2^2 + 1^2)/(2^2 + 2^2)] = 0.9677$$

$$r_{2,0} = r_{2,2}[(2^2 + 0^2)/(2^2 + 2^2)] = 0.7742$$

where,

$$r_{2,2} = 1.5484$$

For the periods

$$T_{2,1} = T_{2,2}(\sqrt{5/8})^3 = 0.9487$$

and

$$T_{2,0} = T_{2,2}(\sqrt{4/8})^3 = 0.6788,$$

where,

$$T_{2,2} = 1.92$$

Table IV. Comparison between the observed results and those calculated from the model.

Position	Planet / asteroid	Mean Radius (AU)		Orbital Period (terrestrial years)	
		Calculated	Observed	Calculated	Observed
n=1;m=1	Mercury	0.387	0.387	0.241	0.241

n=2;m=0	Venus	0.774	0.723	0.682	0.615
n=2;m=1	Earth	0.968	1.0	0.953	1.0
n=2;m=2	Mars	1.548	1.527	1.928	1.881
n=3;m=0	HIL	1.742	1.780	2.30	2.375
n=3;m=1	HOL	1.935	2.0	2.694	2.828
n=3;m=2	Vesta	2.515	2.361	3.994	3.60
n=3;m=3	Camilla	3.483	3.477	6.507	6.502
n=4	Jupiter	6.192	5.203	15.424	11.962
n=5	Saturn	9.675	9.539	30.125	29.458
n=6	Chiron	13.932	13.708	52.056	50.760
n=7	Uranus	18.963	19.191	82.663	84.010
n=8	Nessus	24.768	24.655	123.392	122.420
n=9	Neptune	31.347	30.061	175.689	164.790
n=10	Pluto	38.70	39.529	241.0	248.540
n=11	Makemake	46.827	45.428	320.771	306.186

n=12	2004 OJ14	55.728	55.762	416.448	411.277
n=13	Eris	65.403	67.902	529.477	559.531

If we consider the expression for the mean planetary radius given by:

$$r = [(n^2 + m^2)/2]r_0$$

for all the planetary orbits where r_0 = 0.387AU, the mean radius of Mercury and *m = n,* with the exception of the orbit of Venus with (*n = 2; m = 0*) and that of Earth (*n = 2; m = 1*), that were obtained through the orbit of Mars (*n = 2; m =* 2)*,* and those corresponding to the interior ring (*n = 3; m = 0*) and the exterior ring (*n = 3; m =* 1) of the Hungarian Asteroid Belt and that assocated to the interior ring of the Main Asteroid Belt (*n = 3; m = 2*) which were obtained starting from the corresponding orbit of the exterior ring of this belt (*n = 3; m*=*3*), then the original expression for the calculation of the mean planetary distances remains unaltered and so the values for the mean planetary radius can be determined as shown in Table V.

Table V. Mean planetary radius calculated by the equation $r = [(n^2 + m^2)/2]\ r_0$

Position	$r = [(n^2 + m^2)/2]r_0$	Mean distance in AU	Planet / Asteroid
$n=1;\ m=1$	$r_1 = r_0 = 0.387$	0.387	Mercúrio
$n=2;\ m=0$	$r_2 = [(2^2+0^2)/2]r_0$	0.774	Venus
$n=2;\ m=1$	$r_3 = [(2^2+1^2)/2]r_0$	0.967	Terra
$N=2;\ m=2$	$r_4 = [(2^2+2^2)/2]r_0$	1.548	Marte
$n=3;\ m=0$	$r_5 = [(3^2+0^2)/2]r_0$	1.741	HIL
$n=3;\ m=1$	$r_6 = [(3^2+1^2)/2]r_0$	1.935	HOL
$n=3;\ m=2$	$r_7 = [(3^2+2^2)/2]r_0$	2.515	Vesta
$n=3;\ m=3$	$r_8 = [(3^2+3^2)/2]r_0$	3.483	Camilla
$n=4;\ m=4$	$r_9 = [(4^2+4^2)/2]r_0$	6.192	Jupiter
$n=5;\ m=5$	$r_{10} = [(5^2+5^2)/2]r_0$	9.675	Saturn
$n=6;\ m=6$	$r_{11} = [(6^2+6^2)/2]r_0$	13.932	Chiron
$n=7;\ m=7$	$r_{12} = [(7^2+7^2)/2]r_0$	18.963	Uranus
$n=8;\ m=8$	$r_{13} = [(8^2+8^2)/2]r_0$	24.768	Nessus, Hylonome
$n=9;\ m=9$	$r_{14} = [(9^2+9^2)/2]\ r_0$	31.347	Neptune
$n=10;\ m=10$	$r_{15} = [(10^2+10^2)/2]r_0$	38.700	Pluto
$n=11;\ m=11$	$r_{15} = [(11^2+11^2)/2]r_0$	46.827	Makemake
$n=12;\ m=12$	$r_{15} = [(12^2+12^2)/2]r_0$	55.728	2004 OJ14
$n=13;\ m=13$	$r_{15} = [(13^2+13^2)/2]r_0$	65.403	Eris

Considerations

The state corresponding to the integer n =8 was published in a first theoretical work [1] as an empty state, aproximately at the same time in which two asteroids were discovered in the predicted position of 24.678 AU, between the orbits of Uranus and Neptune, named 1993HA2 (mean radius of 24.76AU) and 1995DW2 (mean radius of 24.17AU). Recently they were named Nessus and Hylonome respectively.

The second and third Kepler laws for the planetary orbits can be verified in the equations for the mean planetary distances and revolution periods determined according to the proposed model.

According to the model, the ratio between the orbital velocities is given by the inverse of the ratio between the integers that label each orbit; this result had already been determined by Kepler.

The theorical mean radius for Jupiter (6.2 AU) differs approximately by 19% of the observed value (5.2 AU). This result can be attributed to the fact that Jupiter contributes with approximately 80% for the total planetary mass, in other words, a strong gravitational attraction effect in relation to the others celestial bodies.

It's important to highlight that the spectrum of numbers *m* for the integers n=2 and n=3 appear only for the planets and asteroids situated between the sun and Jupiter.

This model presupposed an unique empty state, found for the pair (n=1; m=0), which could correspond to the existence of bodies orbiting around the sun every 28 days and at a medium radius of 0.18U.

The model could predict the orbits of the dwarf planets Makemake and Eris, as well the orbit of Pluto.

Conclusions

The theoretical results presented for all the mean planetary distances of the bodies orbiting the Solar System have a single starting parameter which is the mean radius of the orbit of Mercury and do not depend on the masses of these bodies indicating a good accordance with the observed distances by the astronomers for both asteroids and heavy planets such as Neptune, Saturn and Uranus, as well for the dwarf planets Pluto, Makemake and Eris. Therefore, the pythagorean idea about the harmony of celestial spheres seems to be equally present in the beauty of the dance that the electrons do around the atomic nucleus and, in a significant manner, manifesting trough the spectral lines characteristic of the electromagnetic radiations emitted by each atom, the structural unit forming part of the constitution of all elements found in Nature.

Could the set of results above be associated to the discretization or quantization of the inherent geometry of the planetary orbits?

2. An equation similar to that of Schrödinger for describing planetary orbits.

An equation similar to Schrödinger's for a disk can be obtained applying the concepts of quantum mechanics to describe the Solar System situated in a plane involving an attractive field situated in the center of this system associated to a body of mass *M* [2-8]. For a body of mass *m* orbiting around this center, the Schrödinger equation in polar coordinates is given by:

$$-\frac{(g^{*})^{2}}{2\mu}\left(\frac{\partial^{2}\psi}{\partial r^{2}}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^{2}}\frac{\partial^{2}\psi}{\partial\theta^{2}}\right)+V(r)\psi=E_{*}\psi$$

where μ is the reduced mass of bodies of mass M and m and $V(r)$ the potential gravitational interaction between these bodies.

In the above equation, given that the potential V is a function only of the radial variable r, this equation can solved by separating the variables in their radial and angular parts:

$$\psi(r,\theta) = f(r)\Theta(\theta)$$

Similarly to the resolution of the Schrödinger's equation for the atomic case the presented equation has solutions Ψ depending on the numbers *n* e *l* which:

$$n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$$

$$l = 0, 1, 2, \ldots, n - \frac{1}{2}$$

As for the calculation of electronic mean distances, the mean planetary distances (mass peaks) can be obtained by:

$$\overline{r}_{nl} = \int_0^{\infty} \int_0^{2\pi} (\psi_{nl}^{*} r \ \psi_{nl}) \ r \ dr d\theta$$

TABLE VI. Mean distances calculated through an equation similar to Schrödinger's equation for solutions associated to the "states" *1/2 ≤ n ≤ 9/2* e *l = 0, 1 …n-1/2*

Celestial bodies	Pairs (*n, l*)	Calculated mean radius in (AU)
Fundamental radius	(1/2, 0)	0.055
Mercury	(3/2,0); (3/2, 1)	0.387; 0.332

Venus	(5/2, 2)	0.829
Earth	(5/2, 0); (5/2, 1)	1.050; 0.995
Mars	(7/2, 3)	1.548
Hungaria asteroids	(7/2, 0); (7/2, 1); (7/2, 2)	2.046; 1.990; 1.824
Main asteroid belt	(9/2, 0); (9/2, 1); (9/2, 2); (9/2, 3); (9/2, 4);	3.372; 3.317; 3.151; 2.875; 2.488

Source: http://www.johnstonsarchive.net/astro/

In Tables VII and VIII the stable orbits of some celestial bodies have mean radii at an average distance between two calculated mass peaks.

TABLE VII. Mean distances calculated through the solutions of an equation similar to Schrödinger's for funcions associated to the "states" $\boldsymbol{11/2 \leq n \leq 27/2}$ **and** $\boldsymbol{l = 0}$

Celestial Bodies	Calculated mass peaks (n, l; r_{nl})	Calculated radius (AU)	Observed radius (AU)
Jupiter, 2000 OZ21	(11/2, 0; 5.031)	5.031	5.203, 4.867
Hidalgo, 1998 WL34	(11/2, 0; 5.031), 13/2, 0; 7.021)	6.026	5.747, 5822
1998 HO121, 2000 VU2	(13/2, 0; 7.021)	7.021	7.135, 6892
Okyrhoe, 1999 LE31	(13/2, 0; 7.021), (15/2, 0; 9.343)	8.182	8.367, 8.133

Saturn 1999 RG33	(15/2, 0; 9.343)	9.343	9.539, 9.377
Echeclus, Thereus	(15/2, 0; 9.343), (17/2, 0; 11.997)	10.670	10.764, 10.638
Damocles, Elatus	(17/2, 0; 11.997)	11.997	11.843, 11.760
Chiron, 166P/NEAT	(17/2, 0; 11.997), (19/2, 0; 14.982)	13.489	13.710, 13.830
Chariklo, 1996 AR20	(19/2, 0; 14.982)	14.982	15.811, 15.197
Bienor, 1999 JV127	(19/2, 0; 14.882), (21/2, 0; 18.299)	16.640	16.519, 16.724
Uranus, Asbolus	(21/2, 0; 18.299)	18.299	19.191, 18.080
Pholus, Pelion	(21/2, 0; 18.299), (23/2, 0; 21.948)	20.123	20.357, 20.053
1999 HD12, 2002 DH5	(23/2, 0; 21. 948)	21.948	21.322, 22.026
Dioresta, 1995 SN55	(23/2, 0; 21.948), (25/2, 0; 25.929)	25.929	25.131, 24.676
Hylonome, Nessus	(25/2, 0; 25.929)	23.934	23.948, 23.564
2002 CB249, 2002 FY36	(25/2, 0; 25.929), (27/2, 0; 30.241)	28.085	28.421, 28.969

Neptune, 2001 QR322	(27/2, 0; 30.241)	30.241	30.061, 30.302

Source: http://www.johnstonsarchive.net/astro/

TABLE VIII. Mean distances calculated through the solutions of an equation similar to Schrödinger's for the functions associated with the "states" ***$27/2 \leq n \leq 33/2$*** **and** ***$l = 0$***

Celestial Bodies	Calculated mass peaks($n, l; r_{nl}$)	Medium radius calculated (AU)	Observed radius (AU)
Neptune, 2001 QR322	(27/2, 0; 30.241)	30.241	30.061, 30.302
1999 CP133, 2001 XH255	(29/2, 0; 34.885)	34.885	34.857, 34.810
Pluto, 90482 Orcus	(31/2, 0; 39.861)	39.861	39.529, 39.187
Haumea, Varuna	(31/2, 0; 39.861), (33/2, 0; 45.168)	42.515	43.136, 42.806
Makemake, Logos	(33/2, 0; 45.168)	45.168	45.428, 45.074

Source: http://www.johnstonsarchive.net/astro/

The set of results presented in the tables VI, VII and VIII, obtained through the solutions of an equation similar to Schrödinger's, allowed to determine:

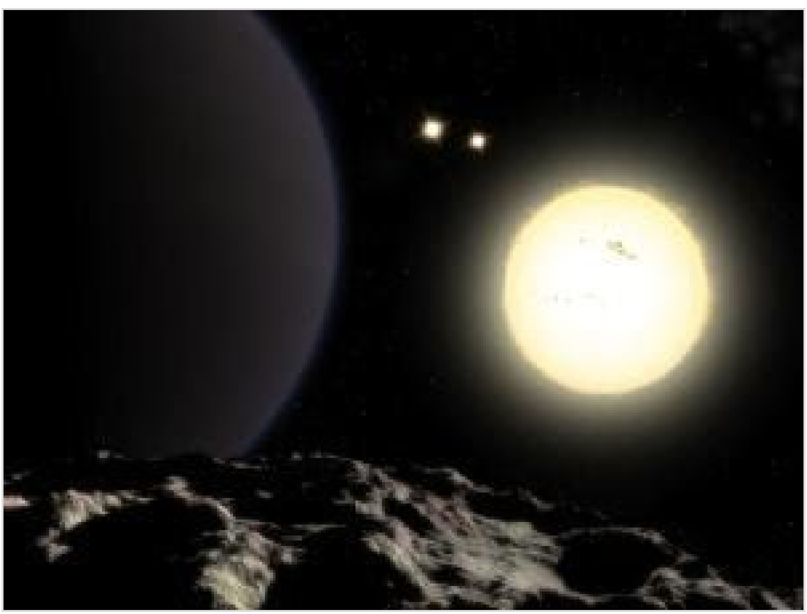
- The value of a "fundamental" radius of 0.05AU, distance where a great number of "hot Jupiter" planets is found in extrasolar systems.
- Mean planetary distances for all the Terrestrial planets (Mercury, Venus, Earth and mars).
- Mean planetary distances for all the Jovian planets (Jupiter, Saturn, Uranus and Neptune).

- Mean planetary distances for all the dwarf planets (Pluto and those recently discovered Makemake, Haumea and Eris).
- Mean radius for the main probability regions where it can be found bodies in the Hungarian and Main Asteroid Belts.
- Mean planetary distances for all the Centaur asteroids and most of the trans-neptunian celestial bodies.

3. A model similar to the molecular quantum mechanics applied to binary star systems

3.1 The extrasolar system HD 188753 Cygni

HD 188753 is an unique extrasolar system in which an unique planet is found orbiting in a triple star system. The planet, a “hot Jupiter” moves around the primary star at a distance of 0.0446 AU (11.5% of the distance between Mercury and the sun). The binary star system is found at a mean separation distance between them of 0.66 AU and 12.3 UA from the primary star. The picture below depicts a representation of this system.



The variational method applied to the H_2^+ molecular ion

The wave function of this system is written as a linear combination of the atomic orbitals *1s* of the hydrogen atom centered in the protons *a* and *b*:

$$\Psi = c_1\psi_{1sa} + c_2\psi_{2sb}$$

The system's energy is given by:

$$E = \frac{\int \Psi^* \hat{H} \Psi d\tau}{\int \Psi^* \Psi d\tau}$$

where H is the hamiltonian operator for the molecular ion:

$$\hat{H} = -\frac{h^2}{8\pi^2 m} + \mathrm{V}$$

Denoting:

$$H_{aa} = H_{bb} = \int \psi_{1Sa}^* \hat{H} \psi_{1Sa} d\tau = \int \psi_{1Sb}^* \hat{H} \psi_{1Sb} d\tau$$
$$H_{ab} = H_{ba} = \int \psi_{1Sa}^* \hat{H} \psi_{1Sb} d\tau == \int \psi_{1Sb}^* \hat{H} \psi_{1Sa} d\tau$$

and

$$S = \int \psi_{1sa}^* \hat{H} \psi_{1sb} d\tau$$

then the expression for the total energy *E* for this system becomes:

$$E = \frac{c_1^2 H_{aa} + 2c_1c_2H_{ab} + c_2^2H_{bb}}{c_1^2 + 2c_1c_2\,S + c_2^2}$$

The minimum of energy in relation to c_1 and c_2 is given by:

$$\frac{\partial E}{\partial c_1} = 0 = c_1(H_{aa} - E) + c_2(H_{ab} - SE)$$

and

$$\frac{\partial E}{\partial c_2} = 0 = c_1(H_{ab} - SE) + c_2(H_{bb} - E)$$

The two non-trivial solutions are obtained by:

$$\begin{vmatrix} H_{aa} - E & H_{ab} - SE \\ H_{ab} - SE & H_{aa} - E \end{vmatrix} = 0$$

The solutions for this determinant are:

$$E_S = \frac{H_{aa} + H_{ab}}{1 + S}$$

and

$$E_A = \frac{H_{aa} - H_{ab}}{1 - S}$$

Replacing these expressions in the above equation for the total energy *E* it can be solved through the ratio c_1/c_2 resulting:

$$c_1 / c_2 = \pm 1$$

leading to the symmetric and antisymmetric solution of the wave function:

$$\Psi_S = c_1(\psi_{1sa} + \psi_{1sb})$$

$$\Psi_A = c_1(\Psi_{1sa} - \psi_{1sb})$$

The constant c_1 can be obtained by the normalization condition of *Ψs* e *Ψa*

$$\int \Psi_S^2 d\tau = 1 \, ; \, \int \Psi_A^2 d\tau = 1$$

or

$$c_1^2[\int \psi_{1Sa}^2 d\tau \pm 2\int \psi_{1Sa}\psi_{1Sb} d\tau + \int \psi_{1Sb}^2 d\tau = 1$$

or

$$c_1^2[1 \pm 2S + 1] = 1$$

So :

$$c_1 = \frac{1}{\sqrt{2 \pm 2S}}$$

The solutions can be given by:

$$\Psi_S = \frac{1}{\sqrt{2+2S}}(\psi_{1Sa} + \psi_{1Sb})$$

and

$$\Psi_A = \frac{1}{\sqrt{2-2S}}(\psi_{1Sa} - \psi_{1Sb})$$

Part of the Hamiltonian operator is the same for the hydrogen atom:

$$-\frac{h^2}{8\pi^2 m}\nabla^2\psi_{1Sa} \quad -\frac{e^2}{r_a}\psi_{1Sa} = E_H\psi_{1Sa}$$

resulting

$$H_{aa} = \int \psi_{1Sa}\left(E_H - \frac{e^2}{r_b} + \frac{e^2}{r_{ab}}\right)\psi_{1Sa}d\tau = E_H + J + \frac{e^2}{a_0 D}$$

J denoting the integral:

$$J = \int \psi_{1Sa}\left(\frac{-e^2}{r_b}\right)\psi_{1Sa}d\tau = \frac{e^2}{a_0}\left[-\frac{1}{D} + e^{-2D}\left(1 + \frac{1}{D}\right)\right]$$

where D = r_{ab}/a_0; a_0 representing the Bohr's radius. In a similar fashion,

$$H_{ab} = \int \psi_{1Sb} \left(E_H - \frac{e^2}{r_b} + \frac{e^2}{r_{ab}} \right) \psi_{1Sa} d\tau = SE_H + K + \frac{Se^2}{a_0 D}$$

$$= SE_H + K + \frac{Se^2}{a_0 D}$$

where

$$S = e^{-D}\left(1 + D + \frac{D^2}{3}\right)$$

and,

$$K = \int \psi_{1Sb}\left(\frac{-e^2}{r_b}\right)\psi_{1Sa} d\tau = \frac{-e^2}{a_0} e^{-D}(1+D)$$

The energies are then given by the following expressions:

$$E_s = E_H + \frac{e^2}{a_0 D} + \frac{J+K}{1+S}$$

and

$$E_A = E_H + \frac{e^2}{a_0 D} + \frac{J-K}{1-S}$$

The stabilization energy is given by the simetric solution E_s of Schrödinger's equation.

Denoting:

$$J' = -\frac{1}{D} + e^{-2D}\left(1 + \frac{1}{D}\right)$$

and

$$K' = -e^{-D}(1 + D)$$

the expression for E_s becomes:

$$E_S = E_H + \frac{e^2}{a_0}\left(\frac{1}{D} + \frac{J' + K'}{1 + S}\right)$$

Defining *E(r)* as:

$$E(r) = \frac{1}{D} + \frac{J' + K'}{1 + S}$$

so

$$E_S = E_H + \frac{e^2}{a_0} E(r)$$

The minimum of energy is given by the equilibrium distance between the two protons.

The variational method allows obtaining a value for the dissociation energy of 1.77 eV for the H_2^+ molecular ion at an equilibrium distance of 1.32 Å. The exact value is 2.78 eV for an equilibrium distance of 1.06 Å.

In the case of the binary stars of the HD 188753 Cygni system if we take the following values for r_{ab} = 0.5 AU; 0.75 AU; 1.0 AU; 1.25 AU e 1.5 AU the results indicate a value around 1.0 AU for the “equilibrium distance” close then to the observed value of 0.66 AU[7].

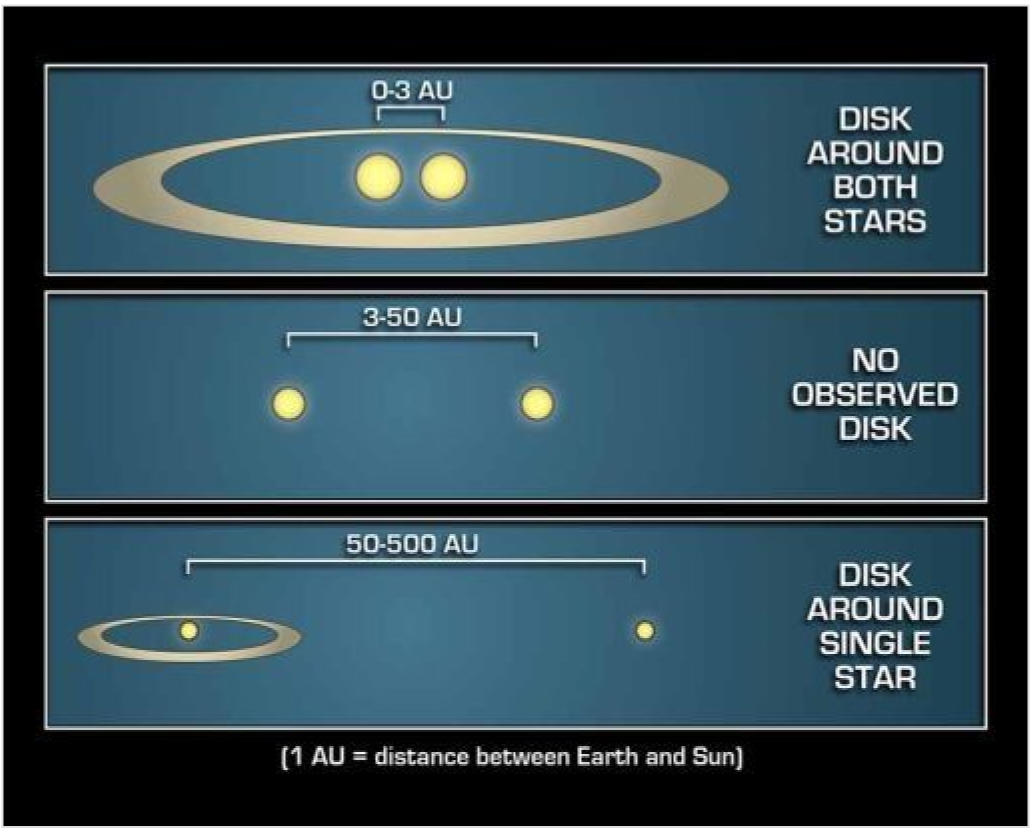
r_{ab}	D	J	K	S	E_S
0,5	1,2920	- 0,6401	- 0,6297	0,7825	- 0,0616
0,75	1,9380	- 0,4846	- 0,4230	0,6033	- 0,050
1,0	2,5840	- 0,3726	- 0,2705	0,4385	- 0,060
1,25	3,2299	- 0,3076	- 0,1673	0,3049	- 0,0543
1,5	3,8760	- 0,2575	- 0,1011	0,2049	- 0,0396

3.2 Geometries of protoplanetary disk formations around a binary star system

Curiously, a recent observation research had as goal to investigate the geometry of protoplanetary disk formation in relation to the separation distances in binary star systems (*Trilling, D. E. et al. (2007). Debris disks in main-sequence binary systems. Astrophys. J. 658, 1289-1311)*. When studying the spectral distribution of energy and typical temperatures (around 100-200K) of dust disks around 69 binary star systems, the results pointed out a few systems in which the dimensions of the dust disks were much bigger than the separation distance betwen the stars implying the existence of circumbinary debris disks with typical separation distances smaller than 3.0 AU. On the other hand, in other binary systems were observed dust distances much smaller than the separation distances between the stars indicating the existence of circumstellar dust disks with typical separation distances bigger than 50.0 AU. In the remaining systems studied the dimensions of the protoplanetary disks were similar to the separation distances between binary stars, indicating instability in the location of these disks.

Apparently, there are clear similarities among the aspects of the chemical bonding obtained by molecular quantum mechanics applied to the system $H_2^+$and the general characteristics of the geometry of formation of protoplanetary disks around binary star systems, particularly in those related to the separation distance between stars. In this sense, the circumbinary debris disks observed at small separation distances of less than 3.0 AU can be considered similar to the formation of stable eletronic clouds around the two nuclei of molecular ion H_2^+ at the equilibrium distance. In a corresponding way, the circumstellar debris disks observed for the binary separations bigger than 50.0 AU are similar to the physical aspects associated to the process of disassociation of the molecular ion H_2^+ resulting in the formation of a hydrogen atom and a proton. Besides, the binary star systems that do not present debris disks circumbinary or circumstellar exhibits unstable electronic cloud geometry. This arrangement is then surprisingly similar to that one found when the two nuclei of the species H_2^+ move away from the equilibrium distance of this molecular ion.

Although the proposed model does not take into account the attractive gravitational potential between the binary stars it seems that the mathematical theoretical framework that describes the geometry of formation of protoplanetary disks in these systems can be similar to the one employed to obtain the electronic distribution functions for the molecular ion H_2^+ in its stable structure as well as in its dissociated form.



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-Planets-Take- Up-Residence.